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On the Matroid Stratification of Grassmann Varieties, Specialization of Coordinates, and a Problem of N. White

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We resolve a problem of N. White [*J. Combin. Theory Ser. B* **29** (1980), 168–175] by constructing representable matroids F and G with $F \leqslant G$ in the weak map order such that no coordinatization of G specializes to a coordinatization of F. Our approach is based upon the interplay of combinatorics and topology in the matroid stratification of the complex Grassmann variety $G_{n,d}$ introduced by I. M. Gelfand, R. M. Goresky, R. D. MacPherson, and V. Serganova [*Adv. in Math.* **63** (1987), 301–316]. Related aspects and open problems in the algebraic geometry of matroids are discussed. © 1989 Academic Press, Inc.

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

The subject of this paper is the algebraic geometry of *matroids* or *combinatorial geometries* in the sense of Crapo and Rota [3] (see also [5, 8, 10, 17]). In contrast to many algebraic situations there is no unique canonical way to define a category with matroids as objects. The chronologically first among two or three competing notions of morphisms between matroids are the so-called *weak maps* [8]. Given two matroids F and G on the same finite set, F is said to be a *weak image* of G, abbreviated $F \leqslant G$, if every basis of F is also a basis of G.

We can think of the points of F as being in a "more special position" than the points of G. Indeed, weak maps are in a certain sense the combinatorial counterpart to the algebraic concept of *specialization of coordinates*. Given a set $M(R)$ of vectors with coordinates in an integral domain R and a prime ideal P of R, consider the induced set of vectors $M(R/P)$ with coordinates in the domain R/P. Then the matroid associated with $M(R/P)$ is a weak image of the matroid associated with $M(R)$ [8, Comm. I.6, p. 48; 17, Exercise 9.2].

In his paper on the transcendence degree of matroid coordinatizations,

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N. White gives the following example [18, Example 3.4] which will also be helpful for our discussion. Let G be the rank 3 matroid on $E = \{1, 2, 3, 4, 5\}$ with non-bases [124] and [135], and let $x_1, ..., x_5$ be algebraically independent over a fixed field K. Consider the representation matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & x_2 & x_4 \\ 0 & 1 & 0 & x_3 & 0 \\ 0 & 0 & x_1 & 0 & x_5 \end{pmatrix}$$

for G over the field $K(x_1, ..., x_5)$. By now succesively setting x_2, x_4, x_3, and x_5 equal to zero, we obtain a sequence of representations of weak images of G, corresponding to an ascending chain of primes in the *bracket ring* B_G^K. The reader is referred to [16] for the definition and fundamental properties of bracket rings, and to Fenton [5] and Bokowski *et al.* [1, 2] for additional background on the algebraic geometry of matroids. Some basic definitions will be given in Section 2.

It is very natural to ask whether every representable weak image F of a representable matroid G can be obtained in the above manner. More precisely, given two matroids $F \leqslant G$, both representable over some field extension of K, can one always find a (transcendental) coordinate matrix for G which specializes to a representation of F? It is proved in [18, Theorem 4.3] that the answer is "yes" if F and G are binary, that is, if $K = \mathbf{GF}_2$. The general case, however, remained open, and N. White suggested the following ring theoretic version of this question.

PROBLEM 1 (White [18, Sect. 3)]. *Given matroids F and G, where $F \leqslant G$ in the weak-map ordering, and each can be coordinatized properly over some extension of K, then do there exist primes p_G and p_F of B_G^K properly coordinatizing G and F (resp.) with $p_G \subset p_F$?*

As a main new result of the present paper we give a negative answer to this question in Section 3 by constructing a counterexample for characteristic zero. Our result is based upon a geometric reformulation of the problem which is, it seems, much closer to intuition. In a very nice paper which appeared recently in this journal Gelfand, Goresky, MacPherson, and Serganova [6] studied various aspects of the matroid stratification Γ of the complex Grassmann variety $G_{n,d}$, i.e., the algebraic variety of d-dimensional vector subspaces of n-dimensional complex space $\mathbf{C}^n$. In Section 2 we outline their approach, and we show how it relates to Problem 1.

Finally, in Section 4 we discuss related results and open questions concerning the algebraic geometry of matroids. In particular, we state a universality result which has been obtained independently by N. E. Mnëv [11] and the author [15]: Every $\mathbf{Q}$-defined affine algebraic $\mathbf{C}$-variety is birationally isomorphic to a stratum of the matroid stratification of $G_{n,3}$ for suitable n. It is a challenging open problem whether these strata are nonsingular varieties. (See the Note Added in Proof.)

2. On the Matroid Stratification of the Complex Grassmann Variety $G_{n,d}$

Throughout this paper we shall deal with the complex numbers $K = \mathbf{C}$ exclusively, with the understanding that most arguments generalize to arbitrary algebraically closed fields. Given integers $n \geqslant d \geqslant 1$, define $\Lambda(n, d) := \{[\lambda_1 \cdots \lambda_d] \mid 1 \leqslant \lambda_1 < \cdots < \lambda_d \leqslant n \text{ and } \lambda_i \in \mathbf{Z}\}$. Let $\mathbf{C}[\Lambda(n, d)]$ denote the complex polynomial ring generated freely by all *brackets* $[\lambda]$, $\lambda \in \Lambda(n, d)$. In this ring we abbreviate $[\lambda_{\pi(1)} \cdots \lambda_{\pi(d)}] := \operatorname{sign} \pi \cdot [\lambda_1 \cdots \lambda_d]$ for any permutation π.

Via Plücker coordinates we view $\mathbf{C}[\Lambda(n, d)]$ as the ring of polynomial functions on the $\binom{n}{d}$-dimensional vector space $\bigwedge_d \mathbf{C}^n$. Write $G_{n,d}$ for the image of the Plücker embedding of the Grassmannian of d-planes in $\mathbf{C}^n$ into the vector space $\bigwedge_d \mathbf{C}^n$. The *Grassmann variety* $G_{n,d}$ consists of all d-vectors $\xi \in \bigwedge_d \mathbf{C}^n$ of the form $\xi = v_1 \vee \cdots \vee v_d$ where $v_1, ..., v_d \in \mathbf{C}^n$.

Let $I_{n,d}$ denote the ideal generated in $\mathbf{C}[\Lambda(n, d)]$ by all quadratic *Grassmann–Plücker syzygies*

$$\sum_{i=1}^{d+1} (-1)^i \cdot [\lambda_1, ..., \lambda_{i-1}, \lambda_{i+1}, ..., \lambda_{d+1}] \cdot [\lambda_i, \mu_1, ..., \mu_{d-1}],$$

where $\lambda \in \Lambda(n, d+1)$, $\mu \in \Lambda(n, d-1)$. By a well-known result of invariant theory (see, e.g., [4, 12, Sect. II.1]), $I_{n,d}$ is precisely the vanishing ideal of $G_{n,d}$ which shows that the Grassmann variety is indeed Zariski closed.

As is customary in algebraic geometry [7, 9] we identify the irreducible subvarieties of $G_{n,d}$ with the prime ideals of its coordinate ring $\mathbf{C}[G_{n,d}] := \mathbf{C}[\Lambda(n, d)]/I_{n,d}$. The complex numbers being algebraically closed, Hilbert's Nullstellensatz [9] implies that every maximal ideal of $\mathbf{C}[G_{n,d}]$ corresponds to a homomorphism $\mathbf{C}[G_{n,d}] \to \mathbf{C}$, i.e., to a point on $G_{n,d}$.

A *representation* over a field extension K of $\mathbf{C}$ of a rank d matroid M on $E = \{1, 2, ..., n\}$ is (in this setup) a homomorphism $\phi: \mathbf{C}[G_{n,d}] \to K$ such that for all $\lambda \in \Lambda(n, d)$: λ is a basis of M if and only if $\phi([\lambda]) \neq 0$. Likewise, ϕ corresponds to a representation of a weak image of M provided $\phi([\lambda]) \neq 0$ implies that λ is a basis of M.

Let I_M denote the ideal in $\mathbf{C}[G_{n,d}]$ which is generated by

$$\{[\lambda],\ \lambda \in \Lambda(n, d) \text{ is dependent in } M\},$$

and let S_M denote the multiplicative semigroup with unit generated by

$$\{[\lambda],\ \lambda \text{ is a basis of } M\}.$$

By standard abuse of notation we write I_M and S_M also for the corresponding sets in polynomial ring $\mathbf{C}[\Lambda(n, d)]$, and we have

Remark 2. Let M be as above, and let K be an extension field of the complex numbers. The representations of M over K are in one-to-one correspondence with the ring homomorphisms $\phi\colon \mathbf{C}[\Lambda(n, d)] \to K$ such that $I_{n,d} + I_M \subset \operatorname{Ker} \phi$ and $S_M \cap \operatorname{Ker} \phi = \varnothing$.

Consider the rings

$$B_M := \mathbf{C}[G_{n,d}]/I_M$$

and

$$R_M := S_M^{-1} \cdot (\mathbf{C}[G_{n,d}]/I_M).$$

B_M is N. White's *bracket ring* of M with coefficients in $\mathbf{C}$. We call the ring R_M the complex *coordinate ring* of M. Note that R_M is isomorphic to Fenton's *simplified Vamos ring* [5]. Remark 2 says that the spectrum $\operatorname{Spec}(R_M)$ of the coordinate ring of M is the space of all representations of M over field extensions of $\mathbf{C}$, and its maximal spectrum $\Gamma_M := \operatorname{Max}(R_M)$ is the space of all complex representations of M (and similarly for the bracket ring B_M and "weak representations of M").

Note that, in general, Γ_M is *not* an (affine) subvariety of $G_{n,d}$ because it is defined by equations *and* inequalities. Its vanishing ideal $\mathscr{I}(\Gamma_M)$ certainly contains the ideal I_M, but, and this will be the crucial point below, in general these two ideals are not equal! In other words, the Zariski closure $\overline{\Gamma_M}$ of Γ_M is not necessarily the weak realization space of M. The set of varieties

$$\Gamma_{n,d} := \{\Gamma_M \mid M \text{ is a complex representable rank } d \text{ matroid on } n \text{ elements}\}$$

partitions $G_{n,d}$ into realization spaces of matroids, and therefore we propose to call $\Gamma_{n,d}$ the *matroid stratification* of the Grassmann variety $G_{n,d}$. This stratification has been studied recently by Gelfand, Goresky, MacPherson, and Serganova [6]. One main new result of their paper is the equivalence of the definition of the matroid stratification given here with the definition of $\Gamma_{n,d}$ as a multi-intersection of all possible Schubert cells with respect to the standard basis of $\mathbf{C}^n$.

In order to get an understanding for the topology of this stratification, it is important to study how the closure $\overline{\Gamma_G}$ of a stratum Γ_G intersects the other strata. In [6, Sect. 5.2] an example, based on harmonic quadrangles, is given of matroids G and F such that

$$\Gamma_F \cap \overline{\Gamma_G} \neq \varnothing \qquad \text{but} \qquad \Gamma_F \not\subset \Gamma_G.$$

In other words, the Grassmann stratification is not a face-to-face cell decomposition of $G_{n,d}$. (From now on we use F and G for matroids to make the connection to White's problem more suggestive.)

We have the following necessary condition for matroids F whose stratum Γ_F intersects the closure of Γ_G.

LEMMA 3. *If $\Gamma_F \cap \overline{\Gamma_G} \neq \varnothing$, then $F \leqslant G$ in the weak map ordering of matroids.*

Proof. Given $\xi \in G_{n,d}$, we write $m_\xi \in \operatorname{Max}(\mathbf{C}[G_{n,d}])$ for the corresponding maximal ideal in the coordinate ring of the Grassmann variety. Assume that $\xi \in \Gamma_F \cap \overline{\Gamma_G}$. The inclusion $\xi \in \Gamma_F$ implies $m_\xi \cap S_F = \varnothing$, while $\xi \in \overline{\Gamma_G}$ implies $m_\xi \supset I_G$. Hence $I_G \cap S_F = \varnothing$, and $[\lambda] \notin I_G$ for every basis λ of F. Therefore every basis of F is also a basis of G. ∎

3. Proof of the Main Result

It is natural to ask whether the converse of Lemma 3 is true, that is, given $F \leqslant G$, both complex coordinatizable, do Γ_F and $\overline{\Gamma_G}$ necessarily intersect? This question turns out to be equivalent to White's Problem 1, and in Theorem 5 we show that the answer to both question is "no."

LEMMA 4. *Let F and G be complex representable* rank *d matroids on $E = \{1, 2, ..., n\}$. Then the following statements are equivalent.*

(a) *There exist primes p_G and p_F of B_G properly coordinatizing G and F (resp.) with $p_G \subset p_F$.*

(b) $\Gamma_F \cap \overline{\Gamma_G} \neq \varnothing$.

Proof. It follows directly from the definition of the bracket ring B_G that statement (a) is equivalent to

(a′) *There exist primes p_G and p_F of $\mathbf{C}[G_{n,d}]$ such that $S_F \cap p_F = \varnothing$, $S_G \cap p_G = \varnothing$, $I_F \subset p_F$, and $I_G \subset p_G \subset p_F$.*

Suppose that (a′) holds. By a lemma of Krull [9, Lemma 4.4] we can assume that p_F is a maximal ideal. By Hilbert's Nullstellensatz there exist a corresponding point $\xi \in G_{n,d}$, and it follows from the properties of p_F in (a′) that $\xi \in \Gamma_F$.

Next observe the representation

$$\mathscr{I}(\Gamma_G) = \bigcap \{p \in \operatorname{Spec}(\mathbf{C}[G_{n,d}]) \mid p \cap S_G = \varnothing \text{ and } p \supset I_G\}. \tag{1}$$

This implies that $p_G \supset \mathscr{I}(\Gamma_G) = \mathscr{I}(\overline{\Gamma_G})$. Since p_G is a subset of p_F, we have $p_F \supset \mathscr{I}(\overline{\Gamma_G})$, and therefore the point ξ which is associated with the maximal ideal p_F is contained in $\overline{\Gamma_G}$.

Conversely, assume that (b) holds. Pick $\xi \in \Gamma_F \cap \overline{\Gamma_G}$, and let $p_F \in \operatorname{Max}(\mathbf{C}[G_{n,d}])$ denote the corresponding maximal ideal. Then $p_F \supset I_F$ and $p_F \cap S_F \neq \varnothing$.

ξ being contained in $\overline{\Gamma_G}$, this is equivalent to $p_F \supset \mathscr{I}(\Gamma_G)$. Equation (1) implies that there exists a prime $p_G \subset p_F$ with $p_G \supset I_G$ and $p_G \cap S_G$. This completes the proof. ∎

Let us express the result of Lemma 4 in more tangible terms. White's Problem 1 is equivalent to the following question:

"*Is there always a realization of F which can be approximated by realizations of G?*" We just have to be a little bit careful about the topologies: If the answer were "yes" in the usual strong topology, then it would also be "yes" in the Zariski topology. But the answer being "no" in the usual strong topology does not automatically imply the same result for the Zariski topology. With a purely ring-theoretic proof we are on the safe side.

THEOREM 5. *There exist complex representable* rank 3 *matroids F and G on $E = \{1, 2, ..., 7\}$ with $F \leqslant G$ such that $\Gamma_F \cap \overline{\Gamma_G} = \varnothing$.*

Proof. Let G be the rank 3 matroid on $E := \{1, 2, ..., 7\}$ with non-bases [147], [257], and [367]. Let F be its weak image with non-bases [124] and $[ij7]$ for all $1 \leqslant i < j \leqslant 6$, that is, 7 is a loop in F (see Fig. 1).

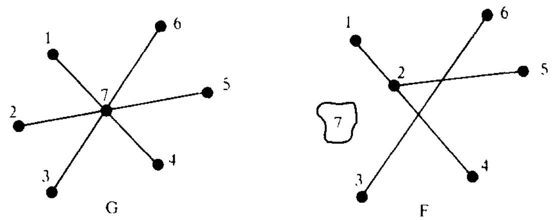

FIG. 1. Two matroids F and G solving N. White's Problem 1.

Consider the polynomial

$$
\begin{aligned}
&+[125][623]\cdot\{[123][147]-[143][127]+[124][173]\}\\
&+[124][125]\cdot\{[123][673]-[623][173]+[163][723]\}\\
&-[124][163]\cdot\{[123][527]-[523][127]+[125][723]\}
\end{aligned}
$$

which is zero in $\mathbf{C}[G_{7,3}]$; for, it is a linear combination of three-term Grassmann–Plücker syzygies $\{\cdots\}$ and thus contained in $I_{7,3}$.

Expanding yields after cancellation of four summands the following representation of the zero in $\mathbf{C}[G_{7,3}]$.

$$
\begin{aligned}
&+[125][623][123]\underline{[147]}\\
&+[124][125][123]\underline{[673]}\\
&-[124][163][123]\underline{[527]}\\
&-[125][623][143][127]\\
&+[124][163][523][127]=0.
\end{aligned}
$$

Since the underlined brackets generate the ideal I_G, we can conclude that $g\cdot[127]\in I_G$ where

$$g:=-[125][623][143]+\underline{[124]}[163][523].$$

By the proof of Lemma 4.3 it is sufficient to show that condition (a′) cannot hold in this situation. Assume on the contrary that there exist primes p_F and p_G as in (a′). Then we have $g\cdot[127]\in p_G$.

Since $[127]\in S_G$, we have $[127]\notin p_G$. And, p_G being a prime ideal, this implies $g\in p_G\subset p_F$.

Since $[124]$ is not a basis of F, we have

$$\underline{[124]}[163][523]\in I_F\subset p_F.$$

Subtracting g which is also contained in the ideal p_F, we obtain $[125][623][143]\in p_F$. On the other hand, this expression is a product of basis brackets with respect to F, and hence $[125][623][143]\in S_F$. Thus we have $p_F\cap S_F\neq\emptyset$, a contradiction. ▌

Our counterexample in Theorem 5 might be somewhat unsatisfactory because the matroid F is not simple, that is, F does not correspond to a geometric lattice. This shortcoming can be mended, however, by embedding both F and G in suitable rank 5 matroids.

4. Related Results and Problems in the Algebraic Geometry of Matroids

The following interesting problem concerning the topology of the Grassmann stratification has been suggested by Gelfand *et al.* [6, Sect. 5.1].

PROBLEM 6. *Given any complex representable matroid M, is its stratum Γ_M necessarily a non-singular variety? In other words, is the ring R_M regular?*

The answer to the same question for the weak realization spaces $\mathrm{Max}(\mathbf{C}[G_{n,d}]/I_M)$ is "no." To see this, it is sufficient to show that for some matroid M the bracket ring B_M is not regular. In [18, Sect. 5] White gives an example of a rank 3 matroid M on 9 elements such that B_M is not Cohen–Macaulay. This implies that B_M is not regular; for, both properties are local, and by a well-known theorem in commutative algebra, every regular Noetherian local ring is Cohen–Macauley [7, Theorem 8.2].

But we can obtain an even stronger result by a more direct argument. By a construction method due to the author [14, 15] which generalizes MacLane's classical arguments in [10], every affine algebraic variety can be encoded as a Zariski dense subset in the realization space (modulo projective transformations) of some rank 3 matroid. With his method it is easy to construct rank 3 matroids M such that $\overline{\Gamma_M}$ has singularities.

However, in all the cases we investigated so far, the singularities occurred *only* because of additional degeneracies, that is, the singular points were always contained in $\overline{\Gamma_M}\backslash\Gamma_M$.

Nevertheless, we conjecture that there exists a matroid M with singularities in Γ_M. First attempts to find such an example indicate that this question seems to be related to the prominent *isotopy conjecture* whether the realization spaces of oriented matroids are necessarily path-connected (see, e.g., [13]). We are convinced that a solution to Problem 6 could give some new insight into the isotopy conjecture as well. But this connection shows also that it is probably very difficult to construct such as "singular" matroid M.

It might be worth mentioning how the encoding procedure in [14; 15, Sect. 2.1] can be translated into the setup of Gelfand, Goresky, MacPherson, and Serganova [6]. The resulting algebraic universality result (Theorem 7) had been found earlier and independently by N. E. Mnëv [11].

The canonical action of the algebraic torus $H := (\mathbf{C}^*)^n$ on $G_{n,d}$ corresponds to the action of the projective group on the vector configurations associated with the points on the Grassmann variety (see [6, Sect. 1.5]). The quotient Γ_M/H of a stratum Γ_M by this algebraic group is the *projective realization space* of M.

THEOREM 7 [11, 14]. *Given any affine algebraic* $\mathbf{C}$*-variety* V *defined over* $\mathbf{Q}$, *there exists an integer* n *and a* rank 3 *matroid* M *on* n *points such that the projective realization space* Γ_M/H *of* M *is birationally isomorphic to* V.

Let us finally remark, that, given a bound $K \in \mathbf{N}$ on the degree and the coefficients of integer polynomials defining V, then the integer n in Theorem 7 can be bounded by a polynomial in K.

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Note added in proof. Problem 6 has been solved by N. E. Mnëv [in O. Y. Viro (Ed.), Topology and Geometry—Rohlin Seminar, Lecture Notes in Mathematics 1346, Springer Verlag, Heidelberg, 1988, pp. 527–544]. The proof of Mnëv's universality theorem implies the existence of complex representable rank matroids M whose stratum Γ_M has singularities.

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